Copositivity for a class of fourth order symmetric tensors given by scalar dark matter

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Abstract. In this paper, we mainly discuss the copositivity of 4th order symmetric tensor defined by scalar dark matter stable under a $\mathbb{Z}_3$ discrete group, and obtain an analytically necessary and sufficient condition of the copositivity of such a class of tensors. Furthermore, this analytic expression may be used to verify the vacuum stability for $\mathbb{Z}_3$ scalar dark matter.

Key Words and Phrases: Homogeneous polynomial, Copositive tensors, symmetric, 4th order Tensors.

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1 Introduction

In particle physics, the standard model of multiple real scalar fields or multiple microscopic particles potentials is fourth-degree homogeneous polynomial. It is well-known that a 4th-degree homogeneous polynomial has a

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one-to-one correspondence with a 4th order symmetric tensor. A physical example is given by scalar dark matter stable under a $\mathbb{Z}_3$ discrete group ([2, 3, 23–25]). That is, the most general scalar potential of a Higgs H_1, an inert doublet H_2 and a complex singlet S is

$$V(h_1, h_2, s) = V(\phi) = \mathcal{V}\phi^4 = \sum_{i,j,k,l=1}^{3} v_{ijkl}\phi_i\phi_j\phi_k\phi_l \tag{1}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T = (h_1, h_2, s)^T$, $h_1 = |H_1|$, $h_2 = |H_2|$, $H_2^\dagger H_1 = h_1 h_2 \rho e^{i\phi}$, $S = se^{i\phi_S}$, $\mathcal{V} = (v_{ijkl})$ is a 4th order 3-dimensional real symmetric tensor. It is obvious that the vacuum stability for $\mathbb{Z}_3$ scalar dark matter (that is, $V(h_1, h_2, s) \geq 0$) is really equivalent to the (strict) copositivity of the tensor $\mathcal{V} = (v_{ijkl})$. The concepts of copositivity and positive definiteness of symmetric tensor were first introduced by Qi [32, 33]. An mth order n-dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is said to be

(i) *(strictly) copositive* if for all non-negative vector $x \in \mathbb{R}^n$ with $\|x\| = 1$,

$$\mathcal{A}x^m = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \geq 0 \ (> 0); \tag{2}$$

(ii) *semipositive (positive) definite* if $\mathcal{A}x^m \geq 0$ (> 0) for all vector $x \in \mathbb{R}^n$ with $\|x\| = 1$ and an even number m.

Qi [33] proved a symmetric tensor is (strictly) copositive if each sum of a diagonal element and all the negative off-diagonal elements in the same row is (positive) nonegative. Subsequently, many good properties are studied for this class of tensors. Song-Qi [37] proposed a method to test the (strict) copositivity of symmetric tensors by using principal sub-tensors. Song-Qi [42] introduced the concepts of Pareto H-eigenvalue and Pareto Z-eigenvalue by means of Lagrange multipliers, and gave the relation between the Pareto H-eigenvalue (Z-eigenvalue) and the (strict) copositivity of corresponding tensor. Song-Qi [40] presented that a symmetric tensor is (strictly) copositive if and only if it is (strictly) semipositive. An mth order n-dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is called (strictly) semipositive if for each non-negative and non-zero vector x, there exists an index $k \in 1, 2, \dots, n$ such that

$$x_k > 0 \text{ and } (\mathcal{A}x^{m-1})_k \geq 0 \ (> 0)$$

This notion is firstly used by Song-Qi [38]. This kind of tensors has good properties and applications in the study of tensor complementary problem. Song-Qi [38] proved every strictly semi-positive tensor is a Q-tensor. Furthermore, we can use it to explore the copositivity of tensors and its applications [39,43,45,47]. More details may be find in refs. [4–7,11,14,15,18,28,49–51] and others.

Besides, some alternative numerical algorithms for copositivity of tensors have been proposed. Chen-Huang-Qi [8] studied some basic theories of copositivity detection of symmetric tensor and presented corresponding numerical algorithms. Li-Zhang-Huang-Qi [26] proposed an SDP relaxation algorithm to test the copositivity of higher order tensors. The more numerical algorithms for checking copositivity of high order tensors were presented by Chen-Huang-Qi [9], Chen-Wang [10]. For more details about copositivity algorithms, also see [30,34,36]. But these conclusions are not really specially designed for the 4th order symmetric tensor, and may not attain the analytic conditions required by the physical problems.

Recently, Song-Qi [41] and Liu-Song [27] respectively presented different sufficient conditions of copositivity. Song [46] presented the positive definiteness of 4th order symmetric tensor. Guo [16] showed a necessary and sufficient condition of a binary quartic form. Very recently, Qi-Song-Zhang [35] showed new necessary and sufficient conditions for quartic polynomial to be positive for all positive reals. Song-Qi [44] gave an analytic necessary and sufficient condition of positive definiteness of 4th order symmetric tensors defined in particle physics. However, the analytic necessary and sufficient conditions have not been obtained for copositivity.

In this paper, we work on seeking analytically checable necessary and sufficient condition for copositivity of 4th order symmetric tensor. Firstly, motivated by Qi-Song-Zhang's [35] result, we show a simple analytical expression of copositivity of 4th order 2-dimensional symmetric tensor. Furthermore, with the help of these conclusions, we discuss copositivity of a 4th order 3-dimensional symmetric tensor defined by vacuum stability for $\mathbb{Z}_3$ scalar dark matter.

2 Preliminaries and Basic facts

It is well-known that both 2×2 matrix (Andersson-Chang-Elfving [1], Hadeler [17], Nadler [31]) and 3×3 matrix (Hadeler [17] and Chang-Sederberg [12])

have analytically copositive condition.

Lemma 2.1. *A real symmetric* 2×2 *matrix* $A = (a_{ij})$ *is (strictly) copositive if and only if*

$$a_{11} \geq 0 \ (> 0), a_{22} \geq 0 \ (> 0), a_{12} + \sqrt{a_{11}a_{22}} \geq 0 \ (> 0).$$

A real symmetric 3×3 *matrix* $A = (a_{ij})$ *is (strictly) copositive if and only if*

$$\begin{aligned} a_{11} \geq 0 \ (> 0), a_{22} \geq 0 \ (> 0), a_{33} \geq 0 \ (> 0), \alpha = a_{12} + \sqrt{a_{11}a_{22}} \geq 0 \ (> 0), \\ \beta = a_{13} + \sqrt{a_{11}a_{33}} \geq 0 \ (> 0), \gamma = a_{23} + \sqrt{a_{33}a_{22}} \geq 0 \ (> 0), \\ a_{12}\sqrt{a_{33}} + a_{13}\sqrt{a_{22}} + a_{23}\sqrt{a_{11}} + \sqrt{a_{11}a_{22}a_{33}} + \sqrt{2\alpha\beta\gamma} \geq 0 \ (> 0). \end{aligned}$$

The non-negativity of a quadratic polynomial with one variable are well-known (also see Qi-Song-Zhang [35] for more details).

Lemma 2.2. *Let f(t) be a quadratic polynomial with one variable and* $a > 0$*,*

$$f(t) = at^2 + bt + c.$$

Then $f(t) > 0 \ (\geq 0)$ *for all* $t \geq 0$ *if and only if*

$$\begin{cases} c > 0 \ (\geq 0), & \text{if } b \geq 0 \\ 4ac - b^2 > 0 \ (\geq 0), & \text{if } b < 0. \end{cases}$$

The non-negativity of a quartic polynomial with one variable is showed by Ulrich-Watson [48] for all positive real numbers. Recently, Qi-Song-Zhang [35] reexpressed their conclusions.

Lemma 2.3. *Let* $f(t)$ *be a quartic polynomial with* $a > 0$ *and* $e > 0$*,*

$$f(t) = at^4 + bt^3 + ct^2 + dt + e.$$

Then $f(t) \geq 0$ *for all* $t > 0$ *if and only if*

(1) $\Delta \leq 0$ *and* $b\sqrt{e} + d\sqrt{a} > 0$*; or*

(2) $b \geq 0$*,* $d \geq 0$ *and* $2\sqrt{ae} + c \geq 0$*; or*

(3) $\Delta \geq 0$*,* $|b\sqrt{e} - d\sqrt{a}| \leq 4\sqrt{ace + 2ae\sqrt{ae}}$ *and either*

(i) $-2\sqrt{ae} \leq c \leq 6\sqrt{ae}$*, or*

(ii) $c > 6\sqrt{ae}$ *and* $b\sqrt{e} + d\sqrt{a} \geq -4\sqrt{ace - 2ae\sqrt{ae}}$*.*

where $\Delta = 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2$*.*

3 Main Results

Let $\mathcal{A}$ be a 4th order 2-dimensional symmetric tensor. Then for a vector $x = (x_1, x_2)^\top$,

$$\begin{aligned}\mathcal{A}x^4 &= \sum_{i,j,k,l=1}^{2} a_{ijkl} x_i x_j x_k x_l \\ &= a_{1111}x_1^4 + 4a_{1112}x_1^3x_2 + 6a_{1122}x_1^2x_2^2 + 4a_{1222}x_1x_2^3 + a_{2222}x_2^4\end{aligned}$$

Next, we give the analytical expression of the copositivity of a 4th order 2-dimensional symmetric tensor.

Theorem 3.1. *Let $\mathcal{A} = (a_{ijkl})$ be a 4th order 2-dimensional symmetric tensor with $a_{1111} > 0$ and $a_{2222} > 0$. Then $\mathcal{A}$ is copositive if and only if*

(1) $I^3 - 27J^2 \leq 0$, $a_{1222}\sqrt{a_{1111}} + a_{1112}\sqrt{a_{2222}} > 0$; *or*

(2) $a_{1222} \geq 0$, $a_{1112} \geq 0$, $3a_{1122} + \sqrt{a_{1111}a_{2222}} \geq 0$; *or*

(3) $I^3 - 27J^2 \geq 0$,
$|a_{1112}\sqrt{a_{2222}} - a_{1222}\sqrt{a_{1111}}| \leq \sqrt{6a_{1111}a_{1122}a_{2222} + 2a_{1111}a_{2222}\sqrt{a_{1111}a_{2222}}}$
(i) $-\sqrt{a_{1111}a_{2222}} \leq 3a_{1122} \leq 3\sqrt{a_{1111}a_{2222}}$;
(ii) $a_{1122} > \sqrt{a_{1111}a_{2222}}$,
$a_{1112}\sqrt{a_{2222}} + a_{1222}\sqrt{a_{1111}} \geq -\sqrt{6a_{1111}a_{1122}a_{2222} - 2a_{1111}a_{2222}\sqrt{a_{1111}a_{2222}}}$,

where $I = a_{1111}a_{2222} - 4a_{1112}a_{1222} + 3a_{1221}^2$,
$J = a_{1111}a_{1122}a_{2222} + 2a_{1112}a_{1122}a_{1222} - a_{1122}^3 - a_{1111}a_{1222}^2 - a_{1112}^2a_{2222}$.

Proof. For $x = (x_1, x_2)^\top$ with $x_i \geq 0$ $(i = 1, 2)$ and $\|x\| = 1$, we have

$$\mathcal{A}x^4 = a_{1111}x_1^4 + 4a_{1112}x_1^3x_2 + 6a_{1122}x_1^2x_2^2 + 4a_{1222}x_1x_2^3 + a_{2222}x_2^4.$$

Obviously, $\mathcal{A}x^4 = a_{2222} > 0$ if $x_1 = 0$ and $x_2 \neq 0$, and $\mathcal{A}x^4 = a_{1111} > 0$ if $x_1 \neq 0$ and $x_2 = 0$. Suppose $x_1 \neq 0$ and $x_2 \neq 0$, we may rewritten the homogeneous polynomial $\mathcal{A}x^4$,

$$\mathcal{A}x^4 = x_1^4(a_{1111} + 4a_{1112}\frac{x_2}{x_1} + 6a_{1122}(\frac{x_2}{x_1})^2 + 4a_{1222}(\frac{x_2}{x_1})^3 + a_{2222}(\frac{x_2}{x_1})^4).$$

Clearly, $\mathcal{A}x^4 \geq 0$ if and only if

$$f(t) = at^4 + bt^3 + ct^2 + dt + e \geq 0,$$

where $a = a_{2222}$, $b = 4a_{1222}$, $c = 6a_{1122}$, $d = 4a_{1112}$, $e = a_{1111}$, $t = \dfrac{x_2}{x_1} > 0$. Then we have

$$\begin{aligned}\Delta &= 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2 \\ &= 4(12a_{1111}a_{2222} - 12 \times 4a_{1112}a_{1122} + 12 \times 3a_{1222}^2)^3 \\ &\quad - (72 \times 6a_{1111}a_{1122}a_{2222} + 72 \times 12a_{1112}a_{1122}a_{1222} - 72 \times 6a_{1122}^3 \\ &\quad - 72 \times 6a_{1112}^2a_{2222} - 72 \times 6a_{1111}a_{1122}^2)^2 \\ &= 4 \times 12^3(a_{1111}a_{2222} - 4a_{1112}a_{1122} + 3a_{1222}^2)^3 \\ &\quad - 72^2 \times 6^2(a_{1111}a_{1122}a_{2222} + 2a_{1112}a_{1122}a_{1222} - a_{1122}^3 - a_{1112}^2a_{2222} - a_{1111}a_{1122}^2)^2 \\ &= 4 \times 12^3(I^3 - 27J^2),\end{aligned}$$

and so, the discriminant Δ and $I^3 - 27J^3$ have the same sign. Then the conclusion may be obtained by Lemma 2.3. □

Now we consider the copositivity of 4th order 3-dimensional symmetric tensor given by scalar dark matter stable under a $\mathbb{Z}_3$ discrete group ([2, 3, 23–25]). The most general scalar quartic potential of the SM Higgs H_1, an inert doublet H_2, and a complex singlet S can be written as

$$\begin{aligned}V(h_1, h_2, s) &= \lambda_1|H_1|^4 + \lambda_2|H_2|^4 + \lambda_3|H_1|^2|H_2|^2 + \lambda_4(H_1^\dagger H_2)(H_2^\dagger H_1) \\ &\quad + \lambda_S|S|^4 + \lambda_{S1}|S|^2|H_1|^2 + \lambda_{S2}|S|^2|H_2|^2 \\ &\quad + \frac{1}{2}(\lambda_{S12}S^2H_1^\dagger H_2 + \lambda_{S12}^* S^{\dagger 2}H_2^\dagger H_1) \\ &= \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 + \lambda_4 \rho^2 h_1^2 h_2^2 \\ &\quad + \lambda_S s^4 + \lambda_{S1}s^2h_1^2 + \lambda_{S2}s^2h_2^2 - |\lambda_{S12}|\rho s^2 h_1 h_2,\end{aligned} \tag{3}$$

where $h_1 = |H_1|$, $h_2 = |H_2|$, $H_1^\dagger H_2 = h_1h_2\rho e^{i\phi}$, $S = se^{i\phi_S}$, $\lambda_{S12} = -|\lambda_{S12}|$, $|\rho| \in [0, 1]$ is the orbit space parameter. Without loss of generality, assuming that $h_1^2 + h_2^2 + s^2 = 1$ in the sequel.

Theorem 3.2. *Let $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_S > 0$ and $\rho_0 = \frac{|\lambda_{S12}|s^2}{2\lambda_4 h_1 h_2}$. Then $V(h_1, h_2, s) \geq 0$ (> 0) for all $h_1 \geq 0$, $h_2 \geq 0$, $s \geq 0$ if and only if $\lambda_{S2} + 2\sqrt{\lambda_2\lambda_S} \geq 0$ (> 0), $\lambda_{S1} + 2\sqrt{\lambda_1\lambda_S} \geq 0$ (> 0) and*

$$\begin{cases}\lambda_3 + \lambda_4 + 2\sqrt{\lambda_1\lambda_2} \geq 0 \ (> 0), & V_{\rho=1}(h_1, h_2, s) \geq 0 \ (> 0), \quad \text{if } \lambda_4 \leq 0 \\ \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0 \ (> 0), & V_{\rho=\rho_0}(h_1, h_2, s) \geq 0 \ (> 0), \quad \text{if } \lambda_4 > 0.\end{cases}$$

Proof. It is obvious that $V(h_1,0,0)=\lambda_1>0$, $V(0,h_2,0)=\lambda_2>0$, $V(0,0,s)=\lambda_S>0$ and

$$V(0,h_2,s)=\lambda_2 h_2^4+\lambda_S s^4+\lambda_{S2}h_2^2s^2=\begin{pmatrix}h_2^2 & s^2\end{pmatrix}\begin{pmatrix}\lambda_2 & \frac{1}{2}\lambda_{S2}\\ \frac{1}{2}\lambda_{S2} & \lambda_S\end{pmatrix}\begin{pmatrix}h_2^2\\ s^2\end{pmatrix}.$$

It follows from Lemma 2.1 that

$$V(0,h_2,s)\geq 0\ (>0) \text{ if and only if } \lambda_{S2}+2\sqrt{\lambda_2\lambda_S}\geq 0\ (>0). \tag{4}$$

Similarly, we also have

$$V(h_1,0,s)\geq 0\ (>0) \text{ if and only if } \lambda_{S1}+2\sqrt{\lambda_1\lambda_S}\geq 0\ (>0), \tag{5}$$

$$V(h_1,h_2,0)\geq 0\ (>0) \text{ if and only if } \lambda_3+\lambda_4\rho^2+2\sqrt{\lambda_1\lambda_2}\geq 0\ (>0).$$

Since $|\rho|\in[0,1]$, the function $f(\rho)=\lambda_3+\lambda_4\rho^2+2\sqrt{\lambda_1\lambda_2}$ reaches its minimum value at $\rho=0$ (if $\lambda_4>0$), $\rho=1$ (if $\lambda_4\leq 0$), and hence,

$$\begin{aligned}V(h_1,h_2,0)\geq 0\ (>0) \text{ if and only if } &\lambda_3+2\sqrt{\lambda_1\lambda_2}\geq 0\ (>0)\\ &\lambda_3+\lambda_4+2\sqrt{\lambda_1\lambda_2}\geq 0\ (>0).\end{aligned} \tag{6}$$

For $h_1>0$, $h_2>0$, $s>0$, we consider the function $g(\rho)=V(h_1,h_2,s)$ about one variable ρ, which is a quadratic function. Clearly,

$$\frac{dg(\rho)}{d\rho}=2\lambda_4\rho h_1^2h_2^2-|\lambda_{S12}|s^2h_1h_2,$$

and so, the function $g(\rho)$ has a unique extremum value at $\rho_0=\frac{|\lambda_{S12}|s^2}{2\lambda_4h_1h_2}$.

If $\lambda_4>0$, then $g(\rho)$ reaches its minimum value at $\rho_0=\frac{|\lambda_{S12}|s^2}{2\lambda_4h_1h_2}$, and hence, $V(h_1,h_2,s)\geq 0$ is now equivalent to $g(\rho_0)\geq 0$. When $\lambda_4\leq 0$, $g(\rho)$ reaches its minimum value at $\rho=1$, and by that time, $V(h_1,h_2,s)\geq 0$ if and only if $g(1)\geq 0$. This completes the proof. □

Let $x=(x_1,x_2,x_3)^T=(h_1,h_2,s)^T$ and

$$\begin{aligned}&v_{1111}=\lambda_1,\ v_{2222}=\lambda_2,\ v_{3333}=\lambda_S,\ v_{1122}=\frac{1}{6}(\lambda_3+\lambda_4),\ v_{1133}=\frac{1}{6}\lambda_{S1},\\ &v_{2233}=\frac{1}{6}\lambda_{S2},\ v_{1233}=-\frac{1}{12}|\lambda_{S12}|,\ v_{ijkl}=0 \text{ for others.}\end{aligned} \tag{7}$$

Then $\mathcal{V} = (v_{ijkl})$ is a 4th order 3-dimensional symmetric tensor and $g(1) = V_{\rho=1}(h_1, h_2, s) = \mathcal{V}x^4$, and so, the inequality $V_{\rho=1}(h_1, h_2, s) \geq 0$ is equivalent to the copositivity of $\mathcal{V}$. By the special structure of $\mathcal{V}$, we now give a necessary and sufficient condition for its copositivity. Let

$$\begin{aligned}
\lambda_{40} &= 4\lambda_S\lambda_1 - \lambda_{S1}^2,\ \lambda_{04} = 4\lambda_S\lambda_2 - \lambda_{S2}^2,\\
\lambda_{13} &= 2\lambda_{S2}|\lambda_{S12}|,\ \lambda_{31} = 2\lambda_{S1}|\lambda_{S12}|,\\
\lambda_{22} &= 4\lambda_S\lambda_3 + 4\lambda_S\lambda_4 - |\lambda_{S12}|^2 - 2\lambda_{S1}\lambda_{S2},\\
\Delta &= 4(12\lambda_{40}\lambda_{04} - 3\lambda_{31}\lambda_{13} + \lambda_{22})^3\\
&\quad - (72\lambda_{40}\lambda_{22}\lambda_{04} + 9\lambda_{31}\lambda_{22}\lambda_{13} - 2\lambda_{22}^3 - 27\lambda_{40}\lambda_{13}^2 - 27\lambda_{31}^2\lambda_{04})^2.
\end{aligned} \tag{8}$$

Theorem 3.3. *Let $\mathcal{V} = (v_{ijkl})$ given by (7) with $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_S > 0$. Then $\mathcal{V}$ is copositive if and only if*

(1) $\lambda_{S1} \geq 0$, $\lambda_{S2} \geq 0$, $2\sqrt{\lambda_{S1}\lambda_{S2}} \geq |\lambda_{S12}|$, $\lambda_3 + \lambda_4 + 2\sqrt{\lambda_1\lambda_2} \geq 0$.

(2) $\lambda_{S1} < 0$, $\lambda_{S2} < 0$, $4\lambda_S\lambda_2 - \lambda_{S2}^2 > 0$, $4\lambda_S\lambda_1 - \lambda_{S1}^2 > 0$ *and*

① $\Delta \leq 0$, $\lambda_{13}\sqrt{\lambda_{40}} + \lambda_{31}\sqrt{\lambda_{04}} > 0$, *or*

② $\Delta \geq 0$, $|\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}$,

(i) $-2\sqrt{\lambda_{40}\lambda_{04}} \leq \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}}$*;*

(ii) $\lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}}$,

$\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}} \geq -4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}$.

Proof. Rewritten the equation (3) as follows,

$$\mathcal{V}x^4 = V(h_1, h_2, s) = \lambda_S t^2 + M(h_1, h_2)t + \widetilde{V}(h_1, h_2),$$

where $t = s^2$,

$$M(h_1, h_2) = \lambda_{S1}h_1^2 + \lambda_{S2}h_2^2 - |\lambda_{S12}|h_1h_2, \tag{9}$$

$$\widetilde{V}(h_1, h_2) = \lambda_1 h_1^4 + \lambda_2 h_2^4 + (\lambda_3 + \lambda_4)h_1^2h_2^2. \tag{10}$$

Then $\mathcal{V}x^4$ can be seen as a one-variable quadratic polynomial about t, and hence, it follows from Lemma 2.2 that $\mathcal{V}$ is copositive if and only if

(1) $M(h_1, h_2) \geq 0$, $\widetilde{V}(h_1, h_2) \geq 0$,

(2) $M(h_1,h_2)<0$, $4\lambda_S\widetilde{V}(h_1,h_2)-(M(h_1,h_2))^2\geq 0$.

Case (1). Clearly, both $M(h_1,h_2)$ and $\widetilde{V}(h_1,h_2)$ are two quadratic form with coefficient matrices

$$\begin{pmatrix} \lambda_{S1} & -\frac{1}{2}|\lambda_{S12}| \\ -\frac{1}{2}|\lambda_{S12}| & \lambda_{S2} \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 & \frac{1}{2}(\lambda_3+\lambda_4) \\ \frac{1}{2}(\lambda_3+\lambda_4) & \lambda_2 \end{pmatrix},$$

and so, it follows from Lemma 2.1 that $M(h_1,h_2)\geq 0$ and $\widetilde{V}(h_1,h_2)\geq 0$ are respectively equivalent to

$$\lambda_{S1}\geq 0,\ \lambda_{S2}\geq 0,\ -|\lambda_{S12}|+2\sqrt{\lambda_{S1}\lambda_{S2}}\geq 0 \text{ and } \lambda_3+\lambda_4+2\sqrt{\lambda_1\lambda_2}\geq 0.$$

Case (2). Obviously, $M(h_1,h_2)<0$ is equivalent to the strict copositivity of $-M(h_1,h_2)$, and then, $M(h_1,h_2)<0$ if and only if

$$\lambda_{S1}<0,\ \lambda_{S2}<0,\ |\lambda_{S12}|+2\sqrt{\lambda_{S1}\lambda_{S2}}>0.$$

It is always tenable that $|\lambda_{S12}|+2\sqrt{\lambda_{S1}\lambda_{S2}}>0$, and hence, $M(h_1,h_2)<0$ if and only if

$$\lambda_{S1}<0,\ \lambda_{S2}<0.$$

Now we prove $4\lambda_S\widetilde{V}(h_1,h_2)-(M(h_1,h_2))^2\geq 0$, which is rewritten as follow,

$$\begin{aligned}
4\lambda_S\widetilde{V}(h_1,h_2)-(M(h_1,h_2))^2 &= 4\lambda_S(\lambda_1h_1^4+\lambda_2h_2^4+(\lambda_3+\lambda_4)h_1^2h_2^2) \\
&\quad -(\lambda_{S1}h_1^2+\lambda_{S2}h_2^2-|\lambda_{S12}|h_1h_2)^2 \\
&= (4\lambda_S\lambda_1-\lambda_{S1}^2)h_1^4+2\lambda_{S1}|\lambda_{S12}|h_1^3h_2 \\
&\quad +(4\lambda_S\lambda_3+4\lambda_S\lambda_4-|\lambda_{S12}|^2-2\lambda_{S1}\lambda_{S2})h_1^2h_2^2 \\
&\quad +2\lambda_{S2}|\lambda_{S12}|h_1h_2^3+(4\lambda_S\lambda_2-\lambda_{S2}^2)h_2^4 \\
&= \lambda_{40}h_1^4+\lambda_{31}h_1^3h_2+\lambda_{22}h_1^2h_2^2+\lambda_{13}h_1h_2^3+\lambda_{04}h_2^4.
\end{aligned}$$

So, this obtain a 4th order 2-dimensional symmetric tensor $\mathcal{A}=(a_{ijkl})$ with

$$a_{1111}=\lambda_{40},\ a_{2222}=\lambda_{04},\ a_{1112}=\frac{1}{4}\lambda_{31},\ a_{1122}=\frac{1}{6}\lambda_{22},\ a_{1222}=\frac{1}{4}\lambda_{13},$$

and then, by Theorem 3.1, we have,

①$\Delta\leq 0$, $\lambda_{13}\sqrt{\lambda_{40}}+\lambda_{31}\sqrt{\lambda_{04}}>0$, or

② $\lambda_{13} \geq 0,\ \lambda_{31} \geq 0,\ \lambda_{22} + 2\sqrt{\lambda_{40}\lambda_{04}} \geq 0$, or

③$\Delta \geq 0,\ |\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}$,

(i)$-2\sqrt{\lambda_{40}\lambda_{04}} \leq \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}}$;

(ii)$\lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}}$,

$$\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}} \geq -4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}.$$

As $\lambda_{S1} < 0$ and $\lambda_{S2} < 0$, there will be no the inequalities $\lambda_{13} = 2\lambda_{S1}|\lambda_{S12}| \geq 0$ and $\lambda_{31} = 2\lambda_{S2}|\lambda_{S12}| \geq 0$, and then, the above conditions ① and ③ guarantee that $4\lambda_S\widetilde{V}(h_1, h_2) - (M(h_1, h_2))^2 \geq 0$. This complete the proof. □

Now we show the necessary and sufficient conditions of $V_{\rho=\rho_0}(h_1, h_2, s) = g(\rho_0) \geq 0$.

Theorem 3.4. *Let $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_S > 0,\ \lambda_4 > 0$ and $\rho_0 = \frac{|\lambda_{S12}|s^2}{2\lambda_4 h_1 h_2}$. Then $V_{\rho=\rho_0}(h_1, h_2, s) \geq 0$ if and only if*

$$4\lambda_4\lambda_S - |\lambda_{S12}|^2 \geq 0,\ \alpha = \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0,$$

$$\beta = \lambda_{S1} + 2\sqrt{\lambda_1(\lambda_S - \tfrac{|\lambda_{S12}|^2}{4\lambda_4})} \geq 0,\ \gamma = \lambda_{S2} + 2\sqrt{\lambda_2(\lambda_S - \tfrac{|\lambda_{S12}|^2}{4\lambda_4})} \geq 0,$$

$$\lambda_3\sqrt{\lambda_S - \tfrac{|\lambda_{S12}|^2}{4\lambda_4}} + \lambda_{S1}\sqrt{\lambda_2} + \lambda_{S2}\sqrt{\lambda_1} + \sqrt{\alpha\beta\gamma} \geq 0.$$

Proof. We plug $\rho_0 = \dfrac{|\lambda_{S12}|s^2}{2\lambda_4 h_1 h_2}$ into the equation (3),

$$\begin{aligned} V_{\rho=\rho_0}(h_1, h_2, s) &= \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_S s^4 + \frac{|\lambda_{S12}|^2}{4\lambda_4}s^4 - \frac{|\lambda_{S12}|^2}{2\lambda_4}s^4 \\ &\quad + \lambda_3 h_1^2 h_2^2 + \lambda_{S1} s^2 h_1^2 + \lambda_{S2} s^2 h_2^2 \\ &= \lambda_1 h_1^4 + \lambda_2 h_2^4 + (\lambda_S - \frac{|\lambda_{S12}|^2}{4\lambda_4})s^4 \\ &\quad + \lambda_3 h_1^2 h_2^2 + \lambda_{S1} s^2 h_1^2 + \lambda_{S2} s^2 h_2^2. \end{aligned} \tag{11}$$

Then $V_{\rho=\rho_0}(h_1, h_2, s)$ may be seen as a quadratic form about (h_1^2, h_2^2, s^2) with the coefficient matrix

$$\begin{pmatrix} \lambda_1 & \frac{1}{2}\lambda_3 & \frac{1}{2}\lambda_{S1} \\ \frac{1}{2}\lambda_3 & \lambda_2 & \frac{1}{2}\lambda_{S2} \\ \frac{1}{2}\lambda_{S1} & \frac{1}{2}\lambda_{S2} & \lambda_S - \frac{|\lambda_{S12}|^2}{4\lambda_4} \end{pmatrix}. \tag{12}$$

Therefore, $V_{\rho=\rho_0}(h_1, h_2, s) \geq 0$ is equivalent to the copositivity of the above coefficient matrix, and hence, after making simple calculations, the desired conclusions directly follow from Lemma 2.1. $\square$

In summary, we obtain an analytical necessary and sufficient condition of copositivity for a special 4th order 3-dimensional symmetric tensor defined by vacuum stability for $\mathbb{Z}_3$ scalar dark matter.

Theorem 3.5. *Let $V(h_1, h_2, s)$ be given by (3) with $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_S > 0$. Then $V(h_1, h_2, s) \geq 0$ if and only if $\lambda_{S2} + 2\sqrt{\lambda_2\lambda_S} \geq 0$, $\lambda_{S1} + 2\sqrt{\lambda_1\lambda_S} \geq 0$ and*

(I) $\lambda_4 > 0$, $4\lambda_4\lambda_S - |\lambda_{S12}|^2 \geq 0$, $\alpha = \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0$,

$$\beta = \lambda_{S1} + 2\sqrt{\lambda_1(\lambda_S - \tfrac{|\lambda_{S12}|^2}{4\lambda_4})} \geq 0,\ \gamma = \lambda_{S2} + 2\sqrt{\lambda_2(\lambda_S - \tfrac{|\lambda_{S12}|^2}{4\lambda_4})} \geq 0,$$

$$\lambda_3\sqrt{\lambda_S - \tfrac{|\lambda_{S12}|^2}{4\lambda_4}} + \lambda_{S1}\sqrt{\lambda_2} + \lambda_{S2}\sqrt{\lambda_1} + \sqrt{\alpha\beta\gamma} \geq 0.$$

(II) $\lambda_4 \leq 0$, $\lambda_3 + \lambda_4 + 2\sqrt{\lambda_1\lambda_2} \geq 0$ *and*

(1) $\lambda_{S1} \geq 0$, $\lambda_{S2} \geq 0$, $2\sqrt{\lambda_{S1}\lambda_{S2}} \geq |\lambda_{S12}|$.

(2) $\lambda_{S1} < 0$, $\lambda_{S2} < 0$, $4\lambda_S\lambda_2 - \lambda_{S2}^2 > 0$, $4\lambda_S\lambda_1 - \lambda_{S1}^2 > 0$ *and*

① $\Delta \leq 0$, $\lambda_{13}\sqrt{\lambda_{40}} + \lambda_{31}\sqrt{\lambda_{04}} > 0$,

② $\Delta \geq 0$, $|\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}$,

(i) $-2\sqrt{\lambda_{40}\lambda_{04}} \leq \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}}$;

(ii) $\lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}}$,

$$\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}} \geq -4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}.$$

Remark 3.1. *An analytically necessary and sufficient conditions of copositivity for a class of special 4th order 3-dimensional symmetric tensors is proved, but the analytical expression of its strict copositivity doesn't still know. Then for a general 4th order 3-dimension symmetric real tensor, how to obtain its analytical expressions of (strict) copositivity.*

Remark 3.2. *For three Higgs doublets with equal electroweak quantum numbers ϕ_i, $i = 1, 2, 3$, the Higgs potential model can construct the following form [13, 19–22, 29],*

$$\begin{aligned}V = &- \frac{M_0}{\sqrt{3}}(\phi_1^*\phi_1 + \phi_2^*\phi_2 + \phi_3^*\phi_3) + \frac{\Lambda_0}{3}(\phi_1^*\phi_1 + \phi_2^*\phi_2 + \phi_3^*\phi_3)^2 \\ &+ \frac{\Lambda_3}{3}\left[(\phi_1^*\phi_1)^2 + (\phi_2^*\phi_2)^2 + (\phi_3^*\phi_3)^2 - (\phi_1^*\phi_1)(\phi_2^*\phi_2) - (\phi_1^*\phi_1)(\phi_3^*\phi_3) - (\phi_2^*\phi_2)(\phi_3^*\phi_3)\right] \\ &+ \Lambda_1\left[(Re\phi_1^*\phi_2)^2 + (Re\phi_2^*\phi_3)^2 + (Re\phi_3^*\phi_1)^2\right] \\ &+ \Lambda_2\left[(Im\phi_1^*\phi_2)^2 + (Im\phi_2^*\phi_3)^2 + (Im\phi_3^*\phi_1)^2\right] \\ &+ \Lambda_4\left[(Re\phi_1^*\phi_2)(Im\phi_1^*\phi_2) + (Re\phi_2^*\phi_3)(Im\phi_2^*\phi_3) + (Re\phi_3^*\phi_1)(Im\phi_3^*\phi_1)\right].\end{aligned}$$

Then how to solve the analytically necessary and sufficient conditions of the boundedness from below of the above model (or $V \geq 0$) is a topic worthy of study and practical significance. It may be seen as a 4th order 3-dimensional symmetric tensors, and so, this problem is converted into a problem that solving positive definiteness (or copositivity) of the corresponding tensor.

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